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Gauge-covariant properties of a linear nonautonomous quantum system: time-dependent even and odd coherent states

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Abstract
A new method for finding the dynamical invariant for a general time-dependent harmonic oscillator is proposed by making use of two linearly independent solutions to the classical equation of motion. It is shown that the dynamical invariant for different gauged Hamiltonians are connected by time-dependent gauge transformations. Therefore, the representation whose bases are the instantaneous eigenstates of the invariant operator is a good representation for the quantum nonautonomous system. In this representation, the wavefunction of the system is gauge covariant and thus any observable physical effect is naturally independent of the choice of gauged Hamiltonians. The exact even and odd coherent states for a time-dependent harmonic oscillator are constructed in terms of these gauge covariant bases. The harmonic oscillator with periodically varying frequency is treated as a demonstration of our general approach.

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1. Introduction

It is well known that the harmonic oscillator (HO) is one of the most important and fundamental objects in physics and mathematical physics. Therefore, it is worthwhile to study the problems related to various generalizations of it [1, 2]. In particular, to include the surrounding influence on a vibration, to simulate the coupling of the vibration with other degrees of freedom, or to describe the quantum motion in a Paul trap [3], one should consider a time-dependent harmonic oscillator (TDHO) with parameters changing in time. Therefore, over the past decade, much attention has been paid to obtain the exact solution of the Schrödinger equation for the TDHO. Several techniques such as the evolution operator method [4], the path-integral method [5], the gauge transformation method or algebraic dynamics [6] etc, have been developed to treat these nonautonomous systems. Since Lewis and Riesenfeld [7] derived a simple relation between the eigenstates of the dynamical invariant and the solutions of the Schrödinger equation, the dynamical invariant method has been extensively employed to investigate various quantum evolution problems of TDHO, e.g., time-dependent coherent states [8] and dynamical squeezing [9] etc. Recently, several authors [9, 10] have constructed a dynamical invariant operator with a complicated form for the TDHO in terms of special solutions with special initial conditions of the classical linear dynamical equation.

As is well known that numerous classical Hamiltonians, which are generalized gauge equivalent to each other, yield one common classical equation of motion [11]. To treat these numerous quantum Hamiltonians corresponding to one classical equation of motion, an energy operator method had been developed [12]. In this approach, one Hamiltonian among the numerous ones was selected as the energy operator. However, energy representation is not an appropriate representation for time-dependent systems since spontaneous transitions among eigenstates of the chosen energy operator take place frequently. Furthermore, it is unclear among so many Hamiltonians which one should serve as the energy operator. Recently, Yeon *et al* [13] have treated the numerous Hamiltonians and the relevant Schrödinger equations in the Hamiltonian representation. However, the relation of the exact solutions for different gauged Hamiltonians is still lacking. In this paper, we show that the dynamical invariant for a TDHO [14], constructed by making use of two linearly independent solutions to the corresponding classical equation of motion, is naturally generalized gauge-covariant and thus the solutions of Schrödinger equations for different gauged Hamiltonians are connected by time-dependent gauge transformations. This is the main distinction of our present approach from quantum canonical transformation methods [15–17], wherein non-unique canonical momentum operators should be treated.

The paper is set out as follows. Section 2 provides a simple approach to construct the dynamical invariant by making use of two arbitrary linearly independent solutions to the classical equation of motion of TDHO. Using this method, we further construct another dynamical invariant for another Hamiltonian describing the same classical dynamical equation of motion as the former one in section 3. We show that the dynamical invariants for different gauged Hamiltonians are really connected by time-dependent gauge transformations. The gauge functions are only dependent on coordinate q and time parameter t. As a consequence, the exact wavefunctions for TDHO in different gauges are generalized gauge-covariant. Some special wavefunctions, e.g. the exact even and odd coherent states (EOCSs) for TDHO, are introduced in this section. As their generalized gauge covariance the physical effects of these states are also independent of the choice of the gauged Hamiltonians. An HO with a periodically varying frequency is treated in section 4 as a demonstration of our general approach. Like the distinction between the CS for TDHO [11, 14] and Gaulber CS for time-independent HO, we show by a numerical method that the quantum statistical properties of EOCSs for TDHO are obviously different from those of the usual EOCSs for time-independent HO. The conclusions are drawn in the final section.

2. Constructing quantum exact solutions from the classical exact solutions

We consider the general TDHO with the following classical equation of motion

$$\dot{q}_{cl} - \frac{\dot{Z}}{Z(t)}\dot{q}_{cl} + [X(t)Z(t)]q_{cl} = 0 \tag{1}$$

which describes a classical harmonic oscillator with time-dependent mass $m(t) = Z^{-1}(t)$ and frequency $\omega(t) = \sqrt{X(t)Z(t)}$. If $Z(t) \equiv 1$ and the frequency is a periodic function of time t, equation (1) reduces to the famous Hill equation. In recent years, a class of solvable Hill equations with two parameters which are continuous periodic functions has been found [18]. For two arbitrary linearly independent solutions $x_1(t)$ and $x_2(t)$ of equation (1), we can easily prove the following identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\{Z^{-1}(t)[x_1(t)\dot{x}_2(t) - x_2(t)\dot{x}_1(t)]\} = 0.$$

Letting

$$\alpha(t) = \ln[x_1(t)x_2(t)] \qquad \beta(t) = \frac{\dot{\alpha}}{2Z(t)} \qquad \gamma = \frac{Z^{-1}(t)}{2}[x_1(t)\dot{x}_2(t) - x_2(t)\dot{x}_1(t)] = \mathrm{const}$$

we can obtain

$$\dot{\beta} + Z(t)\beta^2(t) + X(t) + Z(t)\gamma^2\mathrm{e}^{-2\alpha(t)} = 0 \tag{2}$$

which will be used in the following to construct dynamical invariants.

The Lagrangian $L(q, \dot{q}, t)$ and classical Hamiltonian $H(q, p, t)$ corresponding to the equation of motion (1) assume the following forms:

$$L(q, \dot{q}, t) = \tfrac{1}{2}Z^{-1}(t)\dot{q}^2 - \tfrac{1}{2}X(t)q^2 \qquad H(q, p, t) = \tfrac{1}{2}[Z(t)p^2 + X(t)q^2] \tag{3}$$

where q and p are classical canonical coordinate and momentum. The corresponding quantum Hamiltonian reads

$$\hat{H}(\hat{q}, \hat{p}, t) = \tfrac{1}{2}[Z(t)\hat{p}^2 + X(t)\hat{q}^2]. \tag{4}$$

To obtain the solution of the Schrödinger equation ($\hbar = 1$)

$$\mathrm{i}\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(\hat{q}, \hat{p}, t)|\psi(t)\rangle \tag{5}$$

corresponding to the above Hamiltonian (4), we first construct the dynamical invariant operator $\hat{I}(\hat{q}, \hat{p}, t)$ which satisfies the conditions

$$\hat{I}^{\dagger}(\hat{q}, \hat{p}, t) = \hat{I}(\hat{q}, \hat{p}, t) \qquad \frac{\partial \hat{I}(\hat{q}, \hat{p}, t)}{\partial t} + \mathrm{i}[\hat{H}(\hat{q}, \hat{p}, t), \hat{I}(\hat{q}, \hat{p}, t)] = 0. \tag{6}$$

Using algebraic dynamical method [6], the dynamical invariant operator $\hat{I}(\hat{q}, \hat{p}, t)$ corresponding to the Hamiltonian (4) can be obtained as follows:

$$\hat{I}(\hat{q}, \hat{p}, t) = \tfrac{1}{2}\left\{\mathrm{e}^{\alpha(t)}(\hat{p} - \beta(t)\hat{q})^2 - \gamma^2\mathrm{e}^{-\alpha(t)}\hat{q}^2\right\} = \hat{U}(t)\hat{I}_0(\hat{q}, \hat{p})\hat{U}^{\dagger}(t) \tag{7}$$

where

$$\hat{I}_0(\hat{q}, \hat{p}) = \tfrac{1}{2}[\hat{p}^2 - \gamma^2\hat{q}^2] \qquad \hat{U}(t) = \mathrm{e}^{\frac{\mathrm{i}}{2}\beta(t)\hat{q}^2}\mathrm{e}^{-\frac{\mathrm{i}\alpha(t)}{4}(\hat{p}\hat{q}+\hat{q}\hat{p})}.$$

The instantaneous eigenstates of $\hat{I}(\hat{q}, \hat{p}, t)$ can be obtained by multiplying the eigenstates of $\hat{I}_0(\hat{q}, \hat{p})$ by the unitary operator $\hat{U}(t)$. The eigenvalue problem of $\hat{I}_0(\hat{q}, \hat{p})$ with real γ for real $x_1(t)$ and $x_2(t)$ is equivalent to that of the Hamiltonian of the time-independent HO with imaginary frequency $\omega = \mathrm{i}\gamma$, which was solved by Zhu and Klauder [19]. However, two linearly independent complex conjugate solutions $x(t)$ and $\bar{x}(t)$ of equation (1) can always be selected. Here $\bar{x}(t)$ denotes the complex conjugation of $x(t) = x_1(t) + \mathrm{i}x_2(t)$. As a consequence, $\gamma = \mathrm{i}\kappa$ with real κ and the eigenvalue problem of $\hat{I}_0(\hat{q}, \hat{p})$ is equivalent to that of the Hamiltonian for a time-independent HO with a real frequency. Therefore, the dynamical invariant operator $\hat{I}(\hat{q}, \hat{p}, t)$ corresponding to the Hamiltonian (4) can be rewritten as

$$\hat{I}(\hat{q}, \hat{p}, t) = \tfrac{1}{2}\left\{\mathrm{e}^{\alpha(t)}(\hat{p} - \beta(t)\hat{q})^2 + \kappa^2\mathrm{e}^{-\alpha(t)}\hat{q}^2\right\} = \left[\hat{a}^{\dagger}(t)\hat{a}(t) + \tfrac{1}{2}\right]\kappa \tag{8}$$

where the raising and lowering operators $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ are defined as

$$\begin{aligned}\hat{a}(t) &= \frac{1}{\sqrt{2\kappa}}\left\{\kappa \mathrm{e}^{-\frac{\alpha(t)}{2}}\hat{q} + \mathrm{i}\mathrm{e}^{\frac{\alpha(t)}{2}}[\hat{p} - \beta(t)\hat{q}]\right\} = \hat{U}(t)\hat{a}_0\hat{U}^\dagger(t)\\ \hat{a}^\dagger(t) &= \frac{1}{\sqrt{2\kappa}}\left\{\kappa \mathrm{e}^{-\frac{\alpha(t)}{2}}\hat{q} - \mathrm{i}\mathrm{e}^{\frac{\alpha(t)}{2}}[\hat{p} - \beta(t)\hat{q}]\right\} = \hat{U}(t)\hat{a}_0^\dagger\hat{U}^\dagger(t)\end{aligned} \tag{9}$$

with

$$\hat{a}_0 = \frac{1}{\sqrt{2\kappa}}(\kappa\hat{q} + \mathrm{i}\hat{p}) \qquad \hat{a}_0^\dagger = \frac{1}{\sqrt{2\kappa}}(\kappa\hat{q} - \mathrm{i}\hat{p}) \qquad [\hat{a}(t), \hat{a}^\dagger(t)] = [\hat{a}_0, \hat{a}_0^\dagger] = 1. \tag{10}$$

Solving the instantaneous eigenvalue problem of the dynamical invariant operator (8): $\hat{I}(\hat{q}, \hat{p}, t)|n, t\rangle = \lambda_n|n, t\rangle$, we have

$$\lambda_n = \left(n + \tfrac{1}{2}\right)\kappa \quad (n = 0, 1, 2, \ldots) \qquad |n, t\rangle = \hat{U}(t)|n\rangle \tag{11}$$

where $|n\rangle = [\sqrt{\kappa}/(2^n n!\sqrt{\pi})]^{1/2}\mathrm{e}^{-\xi^2/2}H_n(\xi)$ with $\xi = \sqrt{\kappa}q$, is the eigenstate of the operator $\hat{I}_0(\hat{q}, \hat{p}) = \frac{1}{2}[\hat{p}^2 + \kappa^2\hat{q}^2]$ with the eigenvalue $\lambda_n = \left(n + \frac{1}{2}\right)\kappa$. Following [6] and [7], we obtain the general solution of Schrödinger equation (5) as follows

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} C_n|\psi_n(t)\rangle \qquad |\psi_n(t)\rangle = \mathrm{e}^{\mathrm{i}\theta_n(t)}|n, t\rangle \qquad C_n = \langle n, 0|\psi(0)\rangle \tag{12}$$

where the total phase or Lewis–Riesenfeld (LR) phase $\theta_n(t)$ reads

$$\theta_n(t) = \int_0^t \left\langle n, t'|\mathrm{i}\frac{\partial}{\partial t'} - \hat{H}(\hat{q}, \hat{p}, t')|n, t'\right\rangle \mathrm{d}t' = -\left(n + \frac{1}{2}\right)\kappa\int_0^t \frac{Z(t')}{|x(t')|^2}\mathrm{d}t'. \tag{13}$$

It should be noted that the two linearly independent solutions of the classical equation of motion can be chosen in different ways which yield different parameters α, β and γ. Therefore, one may define different dynamical invariant representations for a common gauged Hamiltonian [20].

3. Gauge covariance of the exact wavefunction for a general TDHO

As is well known, the Lagranians and Hamiltonians for a defined mechanical system are not unique [12, 14] at the classical level. Indeed, for the general TDHO considered here, we can easily find that the new Lagrangian $L'(q, \dot{q}, t)$ expressed by

$$L'(q, \dot{q}, t) = \frac{1}{2}\left[Z^{-1}(t)\dot{q}^2 - X(t)q^2\right] + \frac{\mathrm{d}G(q, t)}{\mathrm{d}t} = L(q, \dot{q}, t) + \frac{\mathrm{d}G(q, t)}{\mathrm{d}t} \tag{14}$$

also yields the classical equation (1). Here $G(q, t)$ is an arbitrary function of coordinate q and time t. The new Hamiltonian corresponding to the Lagrangian $L'(q, \dot{q}, t)$ in equation (14) reads

$$H'(q, p, t) = \frac{1}{2}[Z(t)(p - \partial G(q, t)/\partial q)^2 + X(t)q^2] - \frac{\partial G(q, t)}{\partial t}. \tag{15}$$

It is easily seen that Hamiltonians (15) and (4) are generalized gauge equivalent as they yield the common classical equation of motion (1). Following Yeon *et al* [13] the quantum Hamiltonian and the relevant Schrödinger equation in new gauge read

$$\hat{H}'(\hat{q}, \hat{p}, t) = \hat{W}(t)\hat{H}(\hat{q}, \hat{p}, t)\hat{W}^\dagger(t) - \mathrm{i}\hat{W}(t)\frac{\partial \hat{W}^\dagger(t)}{\partial t} \qquad \hat{W}(t) = \mathrm{e}^{\mathrm{i}G(\hat{q}, t)} \tag{16}$$

and

$$\mathrm{i}\frac{\partial}{\partial t}|\phi(t)\rangle = \hat{H}'(\hat{q}, \hat{p}, t)|\phi(t)\rangle \tag{17}$$

respectively. This means that there also exists a non-unique description of Lagranians and Hamiltonians for a defined quantum system. However, the physical observable of this dynamical system should be independent of the choice of the gauges. Differentiating from the energy operator method [12], we now show this requirement is naturally satisfied by using the gauge-covariant dynamical invariant. It is easy to prove that the dynamical invariant operator corresponding to the Hamiltonian (16) can be written as

$$\hat{I}'(\hat{q},\hat{p},t)=\hat{W}(t)\hat{I}(\hat{q},\hat{p},t)\hat{W}^{\dagger}(t)=\left[(\hat{a}'(t))^{\dagger}\hat{a}'(t)+\tfrac{1}{2}\right]\kappa \tag{18}$$

where

$$\begin{aligned}\hat{a}'(t)&=\frac{1}{\sqrt{2\kappa}}\left\{\kappa\mathrm{e}^{-\frac{\alpha(t)}{2}}\hat{q}+\mathrm{i}\mathrm{e}^{\frac{\alpha(t)}{2}}[\hat{p}-\beta(t)\hat{q}-\partial G(\hat{q},t)/\partial\hat{q}]\right\}=\hat{W}(t)\hat{a}(t)\hat{W}^{\dagger}(t)\\(\hat{a}'(t))^{\dagger}&=\frac{1}{\sqrt{2\kappa}}\left\{\kappa\mathrm{e}^{-\frac{\alpha(t)}{2}}\hat{q}-\mathrm{i}\mathrm{e}^{\frac{\alpha(t)}{2}}[\hat{p}-\beta(t)\hat{q}-\partial G(\hat{q},t)/\partial\hat{q}]\right\}=\hat{W}(t)\hat{a}^{\dagger}(t)\hat{W}^{\dagger}(t)\end{aligned} \tag{19}$$

satisfy the commutation relation $[\hat{a}'(t),(\hat{a}'(t))^{\dagger}]=1$. As a consequence, a simple relation between the instantaneous eigensolutions of $\hat{I}'(\hat{q},\hat{p},t)$: $\hat{I}'(\hat{q},\hat{p},t)|n,t\rangle'=\lambda_n'|n,t\rangle'$ and those of $\hat{I}(\hat{q},\hat{p},t)$ follows:

$$\lambda_n'=\lambda_n \qquad |n,t\rangle'=\hat{W}(t)|n,t\rangle=\hat{W}(t)\hat{U}(t)|n\rangle. \tag{20}$$

Thus the general solutions of Schrödinger equation (17) read

$$|\phi(t)\rangle=\sum_{n=0}^{\infty}C_n|\phi_n(t)\rangle \qquad |\phi_n(t)\rangle=\mathrm{e}^{\mathrm{i}\theta_n'(t)}|n,t\rangle'. \tag{21}$$

With the help of equations (13), (16) and (20), we can easily prove that

$$\begin{aligned}\theta_n'(t)&=\int_0^t{}'\langle n,\tau|\mathrm{i}\frac{\partial}{\partial\tau}-\hat{H}'(\hat{q},\hat{p},\tau)|n,\tau\rangle'\,\mathrm{d}\tau\\&=\int_0^t\langle n,\tau|\mathrm{i}\frac{\partial}{\partial\tau}-\hat{H}(\hat{q},\hat{p},\tau)|n,\tau\rangle\,\mathrm{d}\tau=\theta_n(t).\end{aligned} \tag{22}$$

This means that the total phase is of gauge independence. At the same time, the gauge covariant solutions to the Schrödinger equations in different gauges are related by gauge transformations, i.e.

$$|\phi(t)\rangle=\hat{W}(t)|\psi(t)\rangle. \tag{23}$$

Coherent states for time-independent quantum system have been widely used in various fields of physics. However, the coherent states for the time-dependent quantum system were not constructed until the invariant theory was developed in 1980s, as for the time-dependent quantum system the invariant rather than Hamiltonian representation is a good representation. The exact CSs for a TDHO were usually constructed in terms of the instantaneous eigenstates of the quadratic time-dependent invariants [8, 10, 14]. Recently, using the linear time-dependent invariant, Man'ko *et al* [9] studied the dynamical effects of a time-dependent frequency on the quantum statistical properties of the initial even and odd coherent states. In what follows we introduce the exact EOCSs for the general TDHO in terms of the instantaneous eigenstates of the quadratic time-dependent invariants. These states should be the solutions of the nonstationary Schrödinger equation and thus they are connected by gauge functions in different generalized gauges. The exact CSs for a general TDHO were defined as the instantaneous eigenstates of the time-dependent annihilation ($\hat{a}(t)$ for the gauge with Hamiltonian (4)) [8, 10, 14], the exact EOCSs for a general TDHO with gauged Hamiltonian (4) thus can be similarly defined as the instantaneous eigenstates of the operator $\hat{a}^2(t)$ with the eigenvalue $[z'(t)]^2$ i.e.

$$|z,t\rangle_e=[\cosh(|z|^2)]^{-1/2}\sum_{n=0}^{\infty}\frac{z^{2n}}{\sqrt{(2n)!}}|\psi_{2n}(t)\rangle \tag{24}$$

and

$$|z,t\rangle_o = [\sinh(|z|^2)]^{-1/2}\sum_{n=0}^{\infty}\frac{z^{2n+1}}{\sqrt{(2n+1)!}}|\psi_{2n+1}(t)\rangle \tag{25}$$

respectively, where $z'(t) = z\mathrm{e}^{2\mathrm{i}\theta_0(t)}$ with $z = |z|\mathrm{e}^{\mathrm{i}\varphi}$ is a time-independent complex. We note that the states $|z,t\rangle_{e,o}$ are exact solutions of Schrödinger equation (5). All auxiliary parameters in the solutions are determined simply by a complex solution to the linear classical equation of motion (1). It is also noted that the exact EOCSs (24), (25) reduce naturally to the usual EOCSs for the time-independent HO Hamiltonian $\hat{H}(\hat{q},\hat{p}) = \frac{1}{2}\left[\hat{p}^2 + \omega_0^2\hat{q}^2\right]$, respectively. Indeed, for the HO with a time-independent frequency ω_0, two linearly independent solutions of equation (1) can be expressed as $x_1(t) = \mathrm{e}^{\mathrm{i}t\omega_0}$ and $x_2(t) = \mathrm{e}^{-\mathrm{i}t\omega_0}$. Thus,

$$\alpha(t) = \beta(t) = 0 \qquad \gamma = \mathrm{i}\kappa \qquad \kappa = \omega_0.$$

In this case, the time-dependent creation and annihilation operators (9) reduce naturally to the usual creation and annihilation operators with a time-independent frequency ω_0.

Similarly, the exact EOCSs for the TDHO in the new gauge described by the new Hamiltonian (16) can also be defined as the eigenstates of the operator $\hat{a}'^2(t)$ with the eigenvalue $[z''(t)]^2$. With the help of equations (19) and (21), we have $z''(t) = z'(t)$ and

$$|z,t\rangle_e' = [\cosh(|z|^2)]^{-1/2}\sum_{n=0}^{\infty}\frac{z^{2n}}{\sqrt{(2n)!}}|\phi_{2n}(t)\rangle = \hat{W}(t)|z,t\rangle_e \tag{26}$$

$$|z,t\rangle_o' = [\sinh(|z|^2)]^{-1/2}\sum_{n=0}^{\infty}\frac{z^{2n+1}}{\sqrt{(2n+1)!}}|\phi_{2n+1}(t)\rangle = \hat{W}(t)|z,t\rangle_o. \tag{27}$$

Therefore, the exact EOCSs for a TDHO defined in the gauge-covariant representation are also gauge-covariant naturally.

4. Quantum statistical properties of the exact EOCSs for the TDHO with periodically varying frequency

The exact EOCSs, as some special gauge-covariant wavefunctions, for a generalized TDHO have been introduced in section 3. We now discuss their quantum statistical properties. Without loss of generality, let us consider the typical time-dependent HO whose classical equation of motion is as follows

$$\ddot{q}_{cl} + \omega^2(t)q_{cl} = 0 \qquad \omega(t+T) = \omega(t) \tag{28}$$

where T is the period of the frequency of the oscillator. If the frequency takes the form: $\omega^2(t) = a + 2q\cos(2t)$, then equation (28) is nothing but the Mathieu equation which describes the classical motion of a particle in a Paul trap [3]. The Mathieu equation has been studied extensively and the solutions are expressed in terms of special functions [22]. In practice, these functions are difficult to work with, not only analytically but also numerically. In order to demonstrate our general approach developed in previous sections, we shall treat the TDHO with periodically varying frequency:

$$\omega^2(t) = 1 + \frac{\xi(1-\eta^2)}{(1+\eta\cos[2t])^2} \tag{29}$$

where $\xi = (a-1)^3/[(a-1)^2 - q^2]$ and $\eta = -q/(a-1)$. It is a sufficiently good approximation to the frequency $\omega^2(t) = a + 2q\cos(2t)$, which takes an important role in a Paul trap [3], if

$|q/(a-1)| < 1$ [23]. In particular, the exact compact solutions of equation (28) with the frequency (29) had been found [24],

$$x_1(t) = \sqrt{\frac{1+\eta\cos[2t]}{1+\eta}} \exp\left\{-\mathrm{i}\frac{\sqrt{1+\xi}}{2}\sin^{-1}\left[\frac{\sqrt{1-\eta^2}\sin[2t]}{1+\eta\cos[2t]}\right]\right\} \tag{30}$$

and $x_2(t) = \bar{x}_1(t)$. So, inserting $x_1(t)$ and $x_2(t)$ into (2) and (13), we have

$$\begin{aligned} &\alpha(t) = \ln\left(\frac{1+\eta\cos[2t]}{1+\eta}\right) \qquad \beta(t) = \frac{-\eta\sin[2t]}{1+\eta\cos[2t]} \qquad \gamma = \mathrm{i}\kappa \\ &\kappa = \sqrt{(1-\eta)(1+\xi)/(1+\eta)} \end{aligned} \tag{31}$$

and

$$\theta_n(t) = -\left(n+\frac{1}{2}\right)\kappa\int_0^t \frac{(1+\eta)\,\mathrm{d}t'}{1+\eta\cos[2t']} = (n+1/2)\sqrt{1+\xi}\arctan\left\{\frac{(\eta-1)}{\sqrt{1-\eta^2}}\tan[t]\right\}. \tag{32}$$

To examine the quantum statistical properties of the exact EOCSs for the TDHO, we introduce the basic quadrature operators $\hat{X}_1$ and $\hat{X}_2$ as follows

$$\hat{X}_1 = \frac{\hat{a}_0 + \hat{a}_0^\dagger}{2} = \sqrt{\frac{\kappa}{2}}\hat{q} \qquad \hat{X}_2 = \frac{\mathrm{i}(\hat{a}_0^\dagger - \hat{a}_0)}{2} = \sqrt{\frac{1}{2\kappa}}\hat{p}_k \tag{33}$$

where $\hat{p}_k = Z^{-1}\partial\hat{H}/\partial\hat{p} = \hat{p}$ is the kinetic momentum operator in the gauge with Hamiltonian (4). In the gauge with Hamiltonian (16) the kinetic momentum operator reads $\hat{p}_k' = Z^{-1}\partial\hat{H}'/\partial\hat{p} = \hat{W}(t)\hat{p}\hat{W}^\dagger(t)$. As a consequence, we easily see that the expected values of the basic quadrature operators $\hat{X}_1$, $\hat{X}_2$ are independent of the choice of the gauge Hamiltonians. The commutation relation between them is also gauge invariant. Therefore, any Hamiltonian can be chosen to discuss the quantum statistical properties of the even and odd coherent states for a TDHO defined in the previous section. For the gauge with Hamiltonian (4), the basic quadrature operators $\hat{X}_1$, $\hat{X}_2$ can be rewritten as

$$\hat{X}_1 = \frac{\mathrm{e}^{\alpha(t)/2}}{2}[\hat{a}(t) + \hat{a}^\dagger(t)] \quad \hat{X}_2 = \frac{\mathrm{e}^{\alpha(t)/2}}{2}\left[\left(\frac{\beta(t)}{\kappa} - \mathrm{i}\mathrm{e}^{-\alpha(t)}\right)\hat{a}(t) + \left(\frac{\beta(t)}{\kappa} + \mathrm{i}\mathrm{e}^{-\alpha(t)}\right)\hat{a}^\dagger(t)\right].$$

A state is said to be squeezed with respect to $\hat{X}_j$ $(j = 1, 2)$ if

$$S_j = (\Delta\hat{X}_j)^2 < 1/4. \tag{34}$$

With the help of equations (9), (24) and (25), we can easily calculate the fluctuations $(\Delta\hat{X}_1)^2$ and $(\Delta\hat{X}_2)^2$ in the exact EOCSs for TDHO with the frequency (29),

$$(\Delta\hat{X}_1)_e^2 = \frac{1+\eta\cos[2t]}{1+\eta}\left[\frac{|z|^2}{2}(\cos\vartheta(t) + \tanh|z|^2) + 1/4\right] \tag{35}$$

$$(\Delta\hat{X}_2)_e^2 = |z|^2\left[P(t)\cos\vartheta(t) + \frac{\beta(t)}{\kappa}\sin\vartheta(t)\right] + S(t)[|z|^2\tanh|z|^2 + 1/2] \tag{36}$$

and

$$(\Delta\hat{X}_1)_o^2 = \frac{1+\eta\cos[2t]}{(1+\eta)}\left[\frac{|z|^2}{2}(\cos\vartheta(t) + \coth|z|^2) + 1/4\right] \tag{37}$$

$$(\Delta\hat{X}_2)_o^2 = |z|^2\left[P(t)\cos\vartheta(t) + \frac{\beta(t)}{\kappa}\sin\vartheta(t)\right] + S(t)[|z|^2\coth|z|^2 + 1/2]. \tag{38}$$

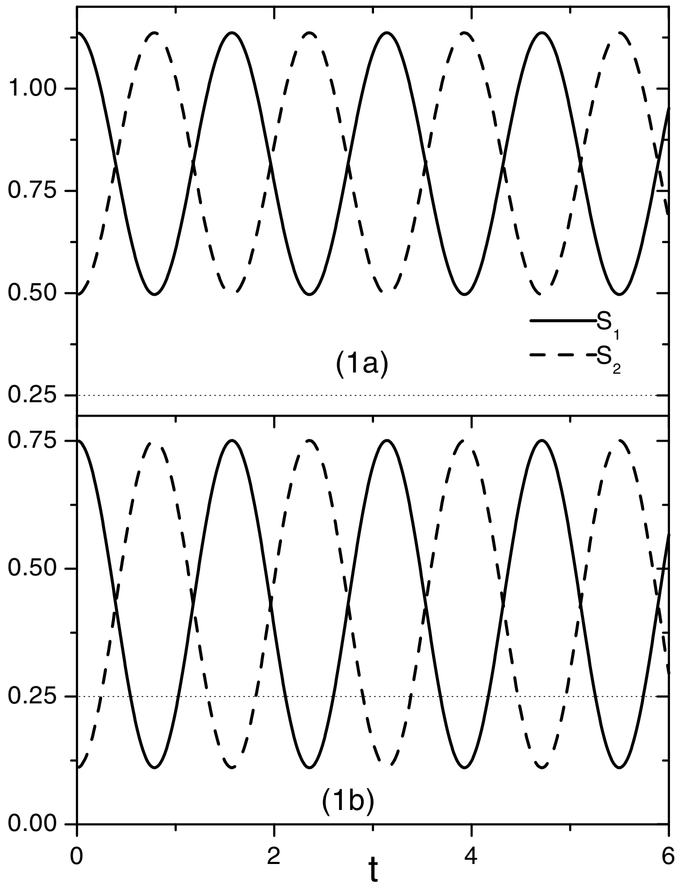

Figure 1. Plots of $S_j = (\Delta\hat{X}_j)^2$ $(j = 1, 2)$ for the exact ECS (b) and the exact OCS (a) versus time t for $z = 0.8$, $\xi = 3$ and $\eta = 0$ (i.e. in the case of the usual time-independent harmonic oscillator). It is shown that for the exact ECS both the quadratures acquire squeezing in periodic time intervals, while for the exact OCS the quadratures do not acquire squeezing at any time.

Here

$$P(t) = \frac{1}{2\kappa}[Q(t)\beta^2(t) - Q^{-1}(t)] \qquad S(t) = \frac{1}{2\kappa}[Q(t)\beta^2(t) + Q^{-1}(t)]$$

$$\vartheta(t) = 2\varphi + 2\sqrt{1+\xi}\arctan\left\{\frac{(\eta-1)}{\sqrt{1-\eta^2}}\tan[t]\right\} \qquad Q(t) = \frac{1+\eta\cos(2t)}{\sqrt{(1-\eta^2)(1+\xi)}}.$$

It is not difficult to see that $\hat{X}_1$ may show squeezing for the exact ECS, but may not for the exact OCS at $t = 0$. In figures 1 and 2, we have plotted the variation of the uncertainties $S_j = (\Delta\hat{X}_j)^2$ versus time t for the exact EOCSs with $\xi = 3, |z| = 0.8, \varphi = 0$ and $\eta = 0$, ± 0.9, respectively. It is noted that for the time-dependent HO both quadratures acquire squeezing in periodic time intervals for the ECS (figure 1(b)), but do not for the OCS (figure 1(a)). We see also that the squeezing of the fluctuation of q (or p_k) appears at the expense of an increase of the fluctuation of p_k (or q). For a TDHO ($\eta = 0.9$) we see that EOCSs possess a time-dependent squeezing on $\hat{X}_1$ (figure 2(a)), while for another TDHO ($\eta = -0.9$) neither ECS nor OCS acquire squeezing on $\hat{X}_2$ (figure 2(b)). These results imply

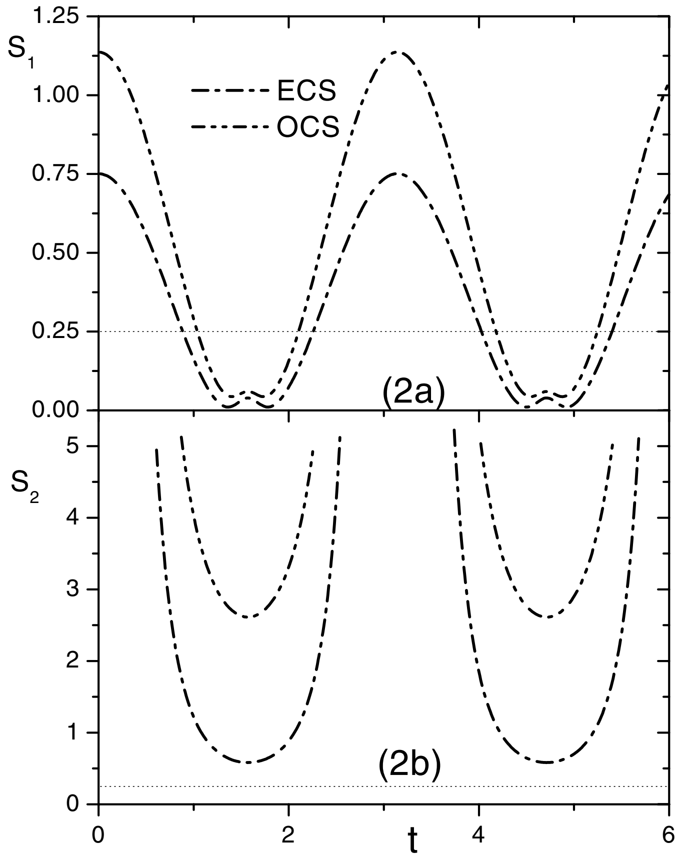

Figure 2. Plots of $S_j = (\Delta \hat{X}_j)^2$ ($j = 1$ or 2) for the exact EOCSs versus time t for $z = 0.8$, $\xi = 3$ and $\eta = 0.9$ (a) or $\eta = -0.9$ (b). Plot (a) shows the quadrature $(\Delta \hat{X}_1)^2$ acquires squeezing in periodic time intervals. Plot (b) shows that the quadrature $(\Delta \hat{X}_2)^2$ does not acquire squeezing at any time.

that the quantum properties of the exact EOCSs for TDHO are quite different from those of the EOCSs for the time-independent HO.

5. Conclusions and discussions

The numerous Hamiltonians and relevant Schrödinger equations, corresponding to a common classical equation of motion, have been treated in the dynamical invariant representation. The new quadratic dynamical invariants for the general TDHO have been constructed by making use of two arbitrary linearly independent solutions of a linear differential equation, which describes the classical dynamics of the general TDHO. It is shown clearly that the invariants for different gauged Hamiltonians are naturally connected by the gauge transformations. Therefore, the instantaneous eigenstates of the dynamical invariants are gauge covariant. The exact solutions of the Schrödinger equation expanded in terms of the instantaneous eigenstates of the dynamical invariant are thus gauge covariant also. Any physical observable should be independent of the choice of gauged Hamiltonians. This physical requirement, which is called gauge independence of physical observables [25], can be best expressed in the dynamical invariant representation. As a demonstration of the general exact wavefunctions of the TDHO, the gauge-covariant exact EOCSs for a general TDHO have been introduced in the invariant representation. They are both the instantaneous eigenstates of the time-dependent annihilation operator with a common time-dependent eigenvalue. The quantum statistical properties of the exact EOCSs for the TDHO periodically varying frequency are discussed in detail by introducing a pair of basic quadrature operators.

Finally, we again point out the differences between our present paper and that in [9], therein the time-dependent EOCSs were defined as the instantaneous eigenstates of the square of the linear time-dependent invariant $\hat{A}(t)$ with changeless eigenvalues. Therefore, these states are in practice the evolutions of the initial usual EOCSs $|\alpha\rangle_{e,o}$ under the time-dependent Hamiltonian. However, just as the exact CSs for the general TDHO were defined as the instantaneous eigenstates of the time-dependent annihilation operator $\hat{a}(t)$, the exact EOCSs for the general TDHO introduced in the present paper were defined as the instantaneous eigenstates of the square of the time-dependent annihilation operator with time-dependent eigenvalues, as the operator $\hat{a}(t)$ is not the dynamical invariant. Of course, $|z, 0\rangle_{e,o}$ is not equal to $|\alpha\rangle_{e,o}$ in general. Therefore, for a common time-dependent Hamiltonian the exact EOCSs we introduced here and the time-dependent EOCSs discussed in [9] reveal different dynamical squeezing. We have also shown that our exact EOCSs reduce to the usual EOCSs if the HO is time-independent.

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